SOME REMARKS ON THE CLASS OF BERNSTEIN FUNCTIONS AND SOME SUB-CLASSES

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1. Introduction

In this paper we summarize some properties that we think complementary to the book recently published of Schilling-Song-Vondracek [4].

We recall the definition of **Bernstein functions** : a function ϕ defined on $[0,+\infty[$ is called a Bernstein function ($\phi \in \mathcal{B}$) if it has the representation

$$\phi(\lambda) = q + d\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\pi(dx), \quad \lambda \geq 0\,, \tag{1.1}$$

q, d are non negative. The measure π is called the Lévy measure of ϕ. It is a measure supported by $]0,\infty[$ that integrates the function $\min(x, 1)$.

Bernstein functions are more likely called by probabilists "Laplace exponents of infinite divisible non negative sub-distributions" (or "Laplace exponents of subordinators") and the previous representation is their Lévy-Khintchine expression.

2. A unified view on subclasses of Bernstein functions

In the sequel all the measures will be understood on the space $]0,\infty[$ and their densities, if they have one, are according to the Lebesgue measure on $]0,\infty[$ which will be denoted by dx.

We recall the definition of the Mellin convolution of two measures ν and τ on the space $]0,\infty[$.

$$\nu \circledast \tau(A) = \int_{]0,\infty[^2} 1_A(xy)\nu(dx)\tau(dy), \quad A\,\text{borelian of}\;]0,\infty[\,.$$

Notice that this integral may be infinite when ν and/or τ are not finite measures.

Theorem 2.1. *Let π be a Lévy measure.*

1) The measure π has a non increasing density if and only if π is of the form

$$\pi = 1_{]0,1]}(x)dx \circledast \nu,$$

where ν is some a Lévy measure.

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2) The measure $x\pi(dx)$ has a non increasing density if and only if π is of the form

$$\pi = 1_{]0,1]}(x)\frac{dx}{x} \circledast \nu,$$

where ν is a measure which integrates the function $f_0(x) = x1_{x\in]0,1]} + \log x 1_{x>1}$ (in particular ν is a Lévy measure).

3) The measure π has a density of the form $x^{a-1}\,k(x)$ with $a \in]-1,\infty[$ and k a completely monotonic function such that $\lim_{+\infty} k(x) = 0$ if and only if π has the expression

$$\pi = x^{a-1}\,e^{-x}dx \circledast \nu,$$

where ν is a measure which integrates the function f_a given by

$$f_a(x) := \begin{cases} x1_{x\in]0,1]} + x^{-a}1_{x>1} & \text{if} \quad a \in]-1,0[, \\ x1_{x\in]0,1]} + \log x 1_{x>1} & \text{if} \quad a = 0, \\ x1_{x\in]0,1]} + 1_{x>1} & \text{if} \quad a \in]0,\infty[\,. \end{cases} \tag{2.1}$$

Consequently, ν is a Lévy measure in all cases. Moreover ν is any Lévy measure in case 1) and in case 3) with $a > 0$.

Proof. Notice that if μ has a density h, then $\mu \circledast \nu$ has the density, denoted by $h \circledast \nu$ and valued in $[0, +\infty]$:

$$h \circledast \nu(x) = \int_{]0,\infty[} \frac{1}{y} h(\frac{x}{y})\nu(dy), \quad x > 0.$$

1) Using the last expression for $h(x) = u_0(x) = 1_{]0,1]}(x)$, we have

$$u \circledast \nu(x) = \int_x^{+\infty} \frac{\nu(dy)}{y}.$$

Notice that any non increasing function (valued in $[0, +\infty]$) is of the form $u_0 \circledast \nu$ and conversely. Since

$$\int_0^\infty (x\wedge 1)u_0 \circledast \nu(x) = \int_0^1 \frac{x^2}{2}\frac{1}{x}\nu(dx) + \frac{1}{2}\int_1^\infty \frac{1}{y}\nu(dy) + \int_1^\infty \frac{x-1}{x}\nu(dx)\,,$$

we deduce that the measure with density $\int_x^{+\infty} \frac{\nu(dy)}{y}$ is a Lévy measure if and only if ν integrates the function $x \wedge 1$ or, in other words, ν is a Lévy measure.

2) Using the expression of $h \circledast \nu$ with $h(x) = u_1(x) = \frac{1}{x}1_{]0,1]}(x)$, we have :

$$u_1 \circledast \nu(x) = \frac{1}{x}\int_x^\infty \nu(dy) = \frac{\nu(]x, +\infty[)}{x}.$$

Notice that any function π, valued in $[0, +\infty]$, such that $x\pi(x)$ is non increasing is of the form $u_1 \circledast \nu$ and conversely. After that, note that

$$\int_0^\infty (x\wedge 1)u_1 \circledast \nu(x)dx = \int_0^1 x\,\nu(dx) + \nu(]1, +\infty[) + \int_1^{+\infty} \log x\,\nu(dx)\,.$$

Thus, the measure with density $u_1 \circledast \nu(x)$ is a Lévy measure if and only if $\nu(dx)$ integrates $f_0(x) = x1_{x<1} + \log x 1_{x>1}$.

3) Let $a \in]-1, \infty[$, k a complete monotone function such that $k(+\infty) =$ and $\pi(dx) = x^{a-1}\, k(x)\, 1_{]0,\infty[}dx$. By Bernstein theorem, k is the Laplace transform of some measure μ on $[0, \infty[$, $k(x) = \int_{[0,\infty[} e^{-xt}\mu(dt)$. Since $k(+\infty) = \mu(\{0\})$, the measure μ does not charge 0. Then, we can build the measure $\Delta(du) = \mu(du)u^{-a}$ and write

$$k(x) := \int_{]0,\infty[} e^{-xu}\; u^a \Delta(du), \quad x > 0\,. \tag{2.2}$$

Let the function

$$h_a(u) := \int_0^\infty (x \wedge u)\, x^{a-1}\, e^{-x} dx, \quad u > 0\,.$$

By Fubini theorem, we have

$$\int_{]0,\infty[} (x \wedge 1)\pi(dx) = \int_{]0,\infty[} (x \wedge 1) x^{a-1}\, k(x) dx = \int_{]0,\infty[} h_a(u) \frac{\Delta(du)}{u}\,.$$

We will find the necessary and sufficient conditions on Δ so that the last integral is finite.

First, notice that $h_a(u) \nearrow \Gamma(a+1)$ when $u \to +\infty$ and then h_a is bounded for any $a > -1$. Second, elementary computations give the following behavior of h_a at the neighborhood of 0 ,

$$\begin{array}{ll} \lim\limits_{0+} \dfrac{h_a(u)}{u} = \Gamma(a) & \text{if} \quad a > 0\,; \\ 0 < \liminf\limits_{0+} \dfrac{h_a(u) - u}{u|\log u|} \leq \limsup\limits_{0+} \dfrac{h_a(u) - u}{u|\log u|} < \infty, & \text{if} \quad a = 0\,; \\ \lim\limits_{0+} \dfrac{h_a(u)}{u^{1-|a|}} = \dfrac{1}{|a|} + \dfrac{1}{1-|a|}, & \text{if} \quad a \in]-1, 0[\,. \end{array}$$

and then π is a Lévy measure iff $\int_1^{+\infty} \frac{\Delta(du)}{u} du < +\infty$ and

$$\Delta([0,1]) < +\infty, \ \text{if} \ \ a > 0, \quad \int_{(0,1]} |\log u|\, \Delta(du) < \infty, \ \text{if} \ \ a = 0, \quad \int_{(0,1]} \frac{\Delta(du)}{u^{|a|}} < \infty, \ \text{if} \ \ -1 < a < 0\,.$$

Notice that in each case $\Delta([0,1]) < +\infty$ and then $\frac{\Delta(du)}{u}$ is a Lévy masure. Define the measure ν as the image of $\Delta(du)$ induced by the function $u \mapsto 1/u$, it is also a Lévy measure. Notice that properties of the measure Δ are equivalent to the fact that the function f_a in the statement of the theorem is integrable for ν. In order to conclude, write

$$x^{a-1}\, k(x) = x^{a-1} \int_{]0,\infty[} e^{-xu}\, u^a\, \Delta(du) = \int_{]0,\infty[} \left(\frac{x}{y}\right)^{a-1} e^{-\frac{x}{y}}\, \frac{\nu(dy)}{y} = [\, y^{a-1} e^{-y} \circledast \nu\,](x)\,.$$

□

Remark 2.2. *Bernstein functions ϕ associated by (1.1) to Lévy measures π of theorem 2.1, are particular in the context of subordinators and infinitely divisible random variables :*

1) A Bernstein function whose Lévy measure is of type (1) is called Jurek Bernstein functions. The Jurek class is also characterized by $\phi \geq 0$ and $(\lambda\phi)' \in \mathcal{B}$.

2) A Bernstein function whose Lévy measure is of type (2) is called self-decomposable Bernstein functions, we denote $\mathcal{SDB}$ their set. It is easy to check (see theorem 2.6 ch. VI [5]*, for instance), that*

$$\phi \in \mathcal{SDB} \quad \Longleftrightarrow \phi(0) \geq 0 \quad \textit{and} \quad \lambda \mapsto \lambda\phi'(\lambda) \in \mathcal{B}.$$

The infinite divisible positive variables X associated to the $\mathcal{SDB}$-functions ϕ ($\mathbb{E}(e^{-\lambda X}) = e^{-\phi(\lambda)}$) are the self-decomposable r.v. X, that is the random variable for which there exists a family of positive r.v. $(Y_c)_{0<c<1}$, each Y_c is independent from X and satisfy $X \stackrel{d}{=} cX + Y_c$.

3) In the book of Schilling-Song-Vondracek the set $\mathcal{CB}$ ([4] p. 49) of Complete Bernstein functions corresponds to the Bernstein functions appearing in part (3) of the above theorem with parameter $a = 1$. The $\mathcal{TB}$ ([4] p. 73) of Thorin Bernstein functions are the ones that with parameter $a = 0$. We will talk again in the next section about CB functions. The class $\mathcal{TB}$ corresponds to the Laplace exponents of the generalized Gamma distributions, shortly GGC, introduced by Bondesson [1], [2] and the GGC subordinators studied by James, Roynette and Yor [3]. For more developments on $\mathcal{TB}$, see [4]. Part 3) of the last theorem suggests a generalization of these two notions for any parameter $a > -1$.

Notice that $\mathcal{CB}_a \subset \mathcal{CB}_b$ for every $a \le b$, $\mathcal{TB} \subset \mathcal{SDB}$ and that $\mathcal{TB} \subset \mathcal{CB}_a \subset \mathcal{CB}$ for every $0 \le a \le 1$.

The simplest functions in $\mathcal{CB}_a$ are given when taking the complete monotonic functions k of the form $k(x) = e^{-bx}$, which is the Laplace transform of δ_b, the Dirac mass at point b. Then the associated Bernstein function is

$$\phi_a(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\, x^{a-1}\, e^{-bx}\, dx = \begin{cases} \Gamma(a)\big(\frac{1}{b^a} - \frac{1}{(b+\lambda)^a}\big) & \text{if} \quad a \neq 0 \\ \log(1 + \frac{\lambda}{b}) & \text{if} \quad a = 0\,. \end{cases} \tag{2.3}$$

Notice that for $a \in]-1, 0[$, these functions are those associated to the so-called tempered stable processes of index $\alpha = -a$ and for $a = 0$, it is the normalized Gamma process. As settled in the next theorem, any $\mathcal{CB}_a$ function is a conic combinations of these simple ones. This result is a straightforward consequence of theorem 2.1:

Theorem 2.3. representation of $\mathcal{CB}_a$-functions. *Let $a > -1$ and ϕ_a defined in (2.3). A function ϕ belongs to $\mathcal{CB}_a$ if and only if its a Mellin convolution of ϕ_a with some measure. Namely, for $a \neq 0$*

$$\phi(\lambda) = q + d\lambda + \int_{]0,\infty[} \Big(1 - \frac{t^a}{(t+\lambda)^a}\Big)\, \nu(dt),$$

where $q, d \ge 0$, ν is a measure that integrates the function $f_a(t)$ given by (2.1). In this case, the Lévy measure associated to ϕ is $\pi(dx) = x^{a-1} \int_0^{+\infty} e^{-xt} t^a \nu(dt)$.

When $a = 0$, we have seen that we meet the well known Thorin class, then we do not recall the facts about that.

3. New injective mappings from $\mathcal{B}$ to $\mathcal{CB}$

We recall that a $\mathcal{CB}$ function is a Bernstein function whose Lévy measure has a density which is a completely monotonic function. We recall the connection between $\mathcal{CB}$ functions; ϕ is a $\mathcal{CB}$ function if and only if it admits the representation

$$\phi(\lambda) = q + a\lambda + \int_{]0,+\infty[} \frac{\lambda}{\lambda + x} \nu(dx) \tag{3.1}$$

where $q, a \ge 0$ and ν is a measure that integrates $\inf(1, 1/x)$.

Another characterization of $\mathcal{CB}$ functions is given by theorem 6.2 of [4] and is known as the **Pick-Nevanlinna characterization of $\mathcal{CB}$ functions :** *Let ϕ a non negative continuous function on $[0, +\infty[$ is a $\mathcal{CB}$ function if and only if it has an analytic continuation on $\mathbf{C} \backslash]-\infty, 0]$ such that $\Im(\phi(z)) \ge 0$ for all z s.t. $\Im(z) > 0$.*

In the next theorem, we will state a representation similar to (3.1) which is valid for any function ϕ in class $\mathcal{B}$.

First, notice that a function ϕ in $\mathcal{B}$ has an analytic continuation on the half plane $\{z, \Re(z) > 0\}$ which can be extended by continuity to the closed half plane $\{z, \Re(z) \geq 0\}$. We still denote ϕ this continuous continuation.

Theorem 3.1. *Let $\phi \in \mathcal{B}$, then, for all $\lambda \geq 0$,*

$$\phi(\lambda) = a\lambda + \frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+u^2}\,\Re[\phi(iu)]\,du\,.$$

Proof. We suppose without loss of generality that $q = a = 0$. Recall $(C_t)_{t\geq 0}$ is a standard Cauchy process. Since $\phi(ix) = \int_{(0,\infty)}(1-e^{-ixs})\pi(ds)$, $x \in \mathbb{R}$, then we can write for all $\lambda > 0$:

$$\begin{aligned}
\phi(\lambda) &= \int_{(0,\infty)} (1-e^{-\lambda s})\,\pi(ds) \\
&= \int_{(0,\infty)} \mathbb{E}[1-e^{-isC_\lambda}]\,\pi(ds) \\
&= \int_{(0,\infty)} \left(\int_{\mathbb{R}} (1-e^{-ius})\frac{\lambda}{\pi(u^2+\lambda^2)}du\right)\pi(ds) \\
&= \int_{\mathbb{R}} \frac{\lambda}{\pi(u^2+\lambda^2)}\,\phi(iu)\,du \\
&= \int_0^\infty \frac{\lambda}{\pi(u^2+\lambda^2)}\,(\phi(iu)+\phi(-iu))du \\
&= \int_0^\infty \frac{\lambda}{\pi(u^2+\lambda^2)}\,2\Re(\phi(iu))du
\end{aligned}$$

□

Notice that $v(u) := \frac{2}{\pi}\Re(\phi(iu))$ is $[0,+\infty[$-valued, negative definite function (see definition 4.3 and theorem 4.6 of [4]) and the identity of the preceding lemma proves that the function $\frac{v(u)}{u^2}$ is integrable at infinity. Denote by $\mathcal{CB}^-$ the class of functions

$$\mathcal{CB}^- := \{\lambda \mapsto \psi(l) = q + d\lambda + \int_0^{+\infty} \frac{\lambda}{\lambda+u^2}\,v(u)\,du, \quad q,\, d \geq 0, v : [0,\infty[\to [0,\infty[\text{ negative definite function}\}$$

It is obvious that $\mathcal{CB}^-$ is a (strict) subclass of $\mathcal{CB}$.

Theorem 3.2. *1) If ϕ is in $\mathcal{B}$, then the function $\lambda \mapsto \sqrt{\lambda}\phi(\sqrt{\lambda})$ is in $\mathcal{CB}^-$, more precisely,*

$$\sqrt{\lambda}\phi(\sqrt{\lambda}) = a\lambda + \int_0^\infty \frac{\lambda}{\lambda+u^2}\,v(u)\,du$$

with $v(u) = \frac{2}{\pi}\Re(\phi(iu))$.

2) Conversely, any function in $\mathcal{CB}^-$ is of the form $\lambda \mapsto \sqrt{\lambda}\phi(\sqrt{\lambda})$ where ϕ is in $\mathcal{B}$.

Proof. Part 1) is given by theorem 3.1. In order to show part 2), suppose v is a (non negative) real valued function, and a negative definite function. Then, according to the Lévy-Khintchine theorem it is the Lévy Fourier exponent of an infinitely divisible distribution and it is symmetric because v is real valued. Necessarily, it is of the form $\Re(\phi(iu))$ where $\phi(iu)$ is a Lévy Fourier exponent of a spectrally positive indefinitely divisible distribution. Now, the integrability of $\frac{v(u)}{u^2}$ at infinity implies that the quadratic component is null and that the Lévy measure integrates $x \wedge 1$ (this is easily seen by the Lévy-Khintchine formula, and can also be found in Vigon thesis [6]). Thus $\phi(iu)$ can be chosen to be the Lévy exponent of a subordinator without a drift. Then, part 1) can be applied to finish the proof. □

Corollary 3.3. *Any Bernstein function leaves globally invariant the cônes* $\{\rho e^{i\pi\alpha};\ \rho \geq 0,\ \alpha \in [-\theta, +\theta]\}$ *for every* $\theta \in [0, \frac{\pi}{2}]$.

Proof. The image of a point λ on the half line $\{\rho e^{i\pi\theta}; \rho \geq 0\}$ by the function $\lambda \mapsto \frac{\lambda}{\lambda^2+u^2}$ (for any $u > 0$), is in the cône $\{\rho e^{i\pi\alpha}; \rho \geq 0,\ \alpha \in [-\theta, +\theta]\}$. This cône is convex and closed, thus, any conic combination of this function, say the integral

$$\frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+u^2}\,\Re(\phi(iu))\,du,$$

is in the same cône. □

Remark 3.4. *This property has to be approached to the much stronger property fulfilled by $\mathcal{CB}$ function : A $\mathcal{CB}$ function has an analytic continuation on $\mathbf{C}\backslash]-\infty, 0]$ and this continuation leaves globally invariant the cônes $\{\rho e^{i\pi\alpha}; \rho \geq 0,\ \alpha \in [0, \theta]\}$ for any $\theta \in [0, 1[$. Moreover this property fully characterizes the $\mathcal{CB}$ functions (the Pick-Nevenlinna characterization given above is equivalent to this property for $\theta = 1$) which is not the case for the property of $\mathcal{B}$- functions given in this corollary.*

In the two following results, we give some extension of theorem 3.2 by replacing the function $\lambda \mapsto \sqrt{\lambda}$ by other functions.

Corollary 3.5. *Let ϕ a $\mathcal{B}$-function.*

1) For any function ψ such that $\lambda \mapsto \psi(\lambda^2)$ is in $\mathcal{B}$, the function $\lambda \mapsto \sqrt{\lambda}\,\phi(\psi(\lambda))$ is in $\mathcal{CB}^-$.

2) For any function ψ such that ψ^2 is in $\mathcal{CB}$, the functions and $\lambda \mapsto \psi(\lambda)\phi(\psi(l))$ and $\frac{\lambda}{\psi(\lambda)}\phi(\psi(\lambda))$ are in $\mathcal{CB}$.

Proof. We notice easily that we can suppose $q = a = 0$ without loss of generality and we do so.

1) Suppose $\lambda \mapsto \psi(\lambda^2)$ is in $\mathcal{B}$, then by composition, $\phi \circ \psi(\lambda^2)$ is a Bernstein function. Theorem 3.1 says that it admits the representation

$$\phi(\psi(\lambda^2)) = \int_0^\infty \frac{\lambda}{\lambda^2+u^2}\,v(u)\,du$$

with v a completely negative function. By change of variables $\lambda \to \sqrt{\lambda}$, we then obtain :

$$\phi(\psi(\lambda)) = \int_0^\infty \frac{\sqrt{\lambda}}{\lambda+u^2}\,v(u)\,du$$

and

$$\sqrt{\lambda}\phi(\psi(\lambda)) = \int_0^{+\infty} \frac{\lambda}{\lambda+u^2}\,v(u)\,du\,.$$

Thus the function $\sqrt{\lambda}\phi(\psi(\lambda))$ is in $\mathcal{CB}^-$.

2) Compose the $\mathcal{CB}$ function $\psi^2(\lambda)$ with the $\mathcal{CB}$ function $\sqrt{\lambda}\phi(\sqrt{\lambda})$ and get the new $\mathcal{CB}$ function $\psi.\,\phi\circ\psi$ (see corollary 7.6 of [4]). This gives the first part of this assertion. Now, take the representation of ϕ of theorem 3.1, and compose with the function ψ, to get

$$\phi(\psi(\lambda)) = \int_0^\infty \frac{\psi(\lambda)}{\psi^2(\lambda)+u^2}\, v(u)\, du, \qquad \Big(\text{with } v(u) = \frac{2}{\pi}(\Re(\phi(iu))\Big).$$

Multiply by $\frac{\lambda}{\psi(\lambda)}$ and obtain

$$\frac{\lambda}{\psi(\lambda)}\phi(\psi(\lambda)) = \int_0^\infty \frac{\lambda}{\psi^2(\lambda)+u^2}\, v(u)\, du\,.$$

Since ψ^2 is in $\mathcal{CB}$, then, so are $\lambda \mapsto \psi^2(\lambda)+u^2$ and $\frac{\lambda}{\psi^2(\lambda)+u^2}$is $\mathcal{CB}$ for every u (see proposition 7.1 of [4]). Thus, the previous integral is a a $\mathcal{CB}$-function as a conic combination of $\mathcal{CB}$-functions.

□

Notice that if f is a $\mathcal{B}$-function, then $f(\sqrt{\lambda})$ satisfies property 1). If further f is a $\mathcal{CB}$-function then $f(\sqrt{\lambda})$ and $\sqrt{f(\lambda)}$ both satisfy property 2).

We summarize now the properties that can be settled when composing a Bernstein function ϕ by a power function $\psi(\lambda) = \lambda^\alpha$, $\alpha \in]0,1]$.

Proposition 3.6. *Let ϕ a function in $\mathcal{B}$,*

1) for all $\alpha \in]0,1]$ the function $\lambda \mapsto \lambda^{1-\alpha}\phi(\lambda^\alpha)$ is in $\mathcal{B}$.

2) If $\alpha \in]0,\frac{1}{2}]$ and $\gamma \in]\alpha, 1-\alpha]$ then $\lambda \mapsto \lambda^\gamma\phi(\lambda^\alpha)$ is in $\mathcal{CB}$. Moreover, if $\gamma = \frac{1}{2}$ it is in $\mathcal{CB}^-$.

Proof. 1) Let $\alpha \in [0,1[$ Denote f_α the density of a normalized α-stable distribution :

$$\int_0^{+\infty} e^{-\lambda x}\, f_\alpha(x)\, dx = e^{-\lambda^\alpha}.$$

Then, for any $t > 0$,

$$e^{-t\lambda^\alpha} = t^{-1/\alpha}\int_0^\infty e^{-\lambda x} f_\alpha(xt^{-1/\alpha})dx\,. \tag{3.2}$$

Let ϕ in $\mathcal{B}$ with Lévy-Khintchine representation:

$$\phi(\lambda) = a + b\lambda + \int_0^{+\infty}(1-e^{-\lambda x})\pi(dx) = a + b\lambda + \lambda.\int_0^{+\infty} e^{-\lambda t}\overline{\pi}(t)dt\,,$$

where π is the Lévy measure and $\overline{\pi}$ its tail function $\overline{\pi}(t) = \pi(]t,+\infty[)$. We have

$$\lambda^{1-\alpha}\phi(\lambda^\alpha) = a + b\lambda + \lambda\int_0^{+\infty} e^{-\lambda^\alpha t}\overline{\pi}(t)dt\,.$$

It is enough to prove that the integral $\int_0^\infty e^{-\lambda^\alpha.t}\overline{\pi}(t)dt$ is the Laplace transform of a non increasing function. For this purpose, we use (3.2) and write

$$\int_0^\infty e^{-\lambda^\alpha t}\overline{\pi}(t)dt = \int_0^\infty t^{-1/\alpha}\left(\int_0^\infty e^{-\lambda z} f_\alpha(zt^{-1/\alpha})dz\right)\overline{\pi}(t)dt\,.$$

By Fubini's theorem, we obtain that the last integral is the Laplace transform at point λ of the function

$$z \mapsto Q(z) := \int_0^\infty f_\alpha(zt^{-\frac{1}{\alpha}})\overline{\pi}(t)t^{-\frac{1}{\alpha}}dt = \int_0^{+\infty} \overline{\pi}(z^\alpha x^{-\alpha})f_\alpha(x)\frac{dx}{x}\,.$$

The second identity comes from the change of variables $t = \frac{z^\alpha}{x^\alpha}$, $x = zt^{-\frac{1}{\alpha}}$. Since the functions $z \mapsto \overline{\pi}(z^\alpha x^{-\alpha})$ are non increasing for all $x > 0$, we deduce that Q is non-increasing.

Now, for $\alpha \leq \frac{1}{2}$, the function $\psi(\lambda) = \lambda^\alpha$ satisfies the properties of corollary 3.5. Part 2) yields that the function $\lambda \mapsto \lambda^{1-\alpha}\phi(\lambda^\alpha)$ is in $\mathcal{CB}$, and part 1), that $\sqrt{\lambda}\phi(\lambda^\alpha)$ is in $\mathcal{CB}^-$.

2) If $\alpha \leq \frac{1}{2}$, then the function $\psi(\lambda) = \lambda^\alpha$ satisfy the part 1) of corollary 3.5 and we get that $\sqrt{\lambda}\phi(\lambda^\alpha)$ is in $\mathcal{CB}^-$.

Let now γ and α s.t. $0 < 2\alpha \leq \gamma + \alpha \leq 1$ and take the representation 3.1 of ϕ in order to obtain,

$$\lambda^\gamma\phi(\lambda^\alpha) = a\lambda^{\gamma+\alpha} + \int_0^\infty \frac{\lambda^{\gamma+\alpha}}{\lambda^{2\alpha} + u^2}v(u)du \qquad v(u) = \frac{2}{\pi}\Re(\phi(iu))$$

Since $0 < 2\alpha \leq \gamma + \alpha \leq 1$, one obtains that the function $\lambda \mapsto \frac{\lambda^{\gamma+\alpha}}{\lambda^{2\alpha}+u^2}$ leaves the half plane $\{\Im(\lambda) > 0\}$ globally invariant. Thus, it is a $\mathcal{CB}$ function for every $u > 0$. By an argument of conic combination, this property remains true for the previous integral and we can conclude that the function $\lambda \mapsto \lambda^\gamma\phi(\lambda^\alpha)$ is $\mathcal{CB}$.

□

4. Three classes of related Bernstein functions

We will discuss about the classes of functions of the form $\lambda \mapsto \phi(\lambda^\alpha)$, and of the form $\lambda \mapsto \phi^\alpha(\lambda)$ and of the form $\lambda \mapsto \int_0^\infty (\frac{\lambda}{\lambda+x})^\alpha$ for any real parameter α. This part will be on line later.

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